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Geometric regularizations and dual conifold transitions

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Abstract: We consider a geometric regularization for the class of conifold transitions relating D-brane systems on noncompact Calabi-Yau spaces to certain flux backgrounds. This regularization respects the $\mathrm{SL}(2, \mathbb{Z})$ invariance of the flux superpotential, and allows for computation of the relevant periods through the method of Picard-Fuchs equations. The regularized geometry is a noncompact Calabi-Yau which can be viewed as a monodromic fibration, with the nontrivial monodromy being induced by the regulator. It reduces to the original, non-monodromic background when the regulator is removed. Using this regularization, we discuss the simple case of the local conifold, and show how the relevant field-theoretic information can be extracted in this approach.

Keywords: Superstring Vacua, Supersymmetric Effective Theories, Supersymmetry and Duality.

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Contents

1. Introduction

Realizing supersymmetric gauge theories in string theory with the help of D-branes has lead to progress in understanding their non-perturbative dynamics. A recent prominent example is $\mathcal{N}=1$ U(N) gauge theory with an additional chiral multiplet in the adjoint representation and a superpotential of the form [1, 2]:

$$W(\Phi) = \sum_{k=1}^{n+1} \frac{t_k}{k} \operatorname{tr}(\Phi^k)\,. \tag{1.1}$$

Based on the results in [3] it has been argued that this can be geometrically engineered in type IIB string theory with the help of the non-compact Calabi-Yau space:

$$W'(x)^2 + f_{n-1}(x) + y^2 + s^2 + t^2 = 0\,, \tag{1.2}$$

where W is the function defined by the superpotential and f_{n-1} is a polynomial of degree $n-1$. When turning off these deformations, the space (1.2) acquires singular points sitting above the n roots x_i of $W'(x) = 0$. The gauge theory interpretation arises by blowing up these singularities to $\mathbb{P}^1$'s and partially wrapping N_i D5-branes on them. The low energy dynamics on the D-branes describes confining vacua of the $\mathcal{N}=1$ supersymmetric gauge theory. This is reflected in a geometric transition where the resolved geometry is replaced by the deformed one (1.2). The D-branes disappear in this process, being replaced by three-from flux through the three-cycles of the deformed geometry. The gauge theory has vacua in which the vev of Φ is $\langle\Phi\rangle = \mathrm{diag}(x_1\mathbf{1}_{N_1},\ldots,x_n\mathbf{1}_{N_n})$ and the gauge group is broken to the product $\prod_{i=1}^n \mathrm{U}(N_i)$. At low energies the non-abelian parts of the broken gauge group will confine and for each gauge group factor there will be a gaugino condensate S_i. The effective superpotential for these condensates can be computed from the Calabi-Yau geometry with the help of its periods:

$$S_i = \int_{A_i}\Omega\,, \qquad \Pi_i = \frac{\partial F}{\partial S_i} = \int_{B_i}\Omega\,, \tag{1.3}$$

where Ω is the holomorphic three form of the Calabi-Yau space and A_i, B_i give a canonical basis of 3-cycles. This superpotential takes the form [4]–[7]:

$$-\frac{1}{2\pi i}W_{\mathrm{eff}} = \sum_{i=1}^{n}(N_i\Pi_i + \alpha_i S_i)\,, \tag{1.4}$$

where N_i, α_i are the fluxes of the type IIB three form $H_R + \tau H_{NS}$ through A_i and B_i. The gauge theory interpretation identifies N_i with the rank of the i^{th} factor of the unbroken gauge group and α_i with the bare coupling of this factor group. The n coefficients of the polynomial $f_{n-1}(x)$ can be determined in terms of the gaugino condensates S_i. Finally, by integrating over s, t one can reduce [8] the Calabi-Yau space to the affine algebraic curve:

$$y^2 + W'(x)^2 + f_{n-1}(x) = 0\,, \tag{1.5}$$

while the holomorphic three-form descends to the following meromorphic differential on this Riemann surface:[1]

$$\omega = \frac{i}{2}y\,dx\,. \tag{1.6}$$

Recently this string theoretic setup has lead to the conjecture that the effective superpotential of the gauge theory can be computed by using a matrix model whose action is given by the tree-level superpotential. The Riemann surface (1.5) has a matrix model interpretation as a spectral curve [11]–[18].

One of the interesting features of this construction is that in the geometry (1.2) only the A-cycles are compact. The 'B-cycles' are non-compact,[2] a feature which forces one to introduce a cutoff Λ_0 in the B-period integrals [2, 9]. Related to this is the fact that the meromorphic form (1.6) is a differential of the third kind, having a nonzero residue at $x = \infty$. This is necessary since otherwise the number of independent A-periods would be only $n-1$, in disagreement with the number of gaugino condensates. Also notice that this regularization breaks the $\mathrm{SL}(2,\mathbb{Z})$ invariance of the flux superpotential (1.4).

For the gauge theory interpretation the cutoff regularization is not unwellcome, since it is used to renormalize the bare gauge couplings α_i. However, it is interesting to ask what

[1] The papers [2, 9, 10] use a different convention for ω, which amounts to dropping a prefactor of $1/2$ at the price of integrating over only half of the length of each Riemann surface cycle. In this paper, we shall always integrate along the full cycles on the Riemann surface.

[2] More precisely, only $n-1$ B-cycles can be chosen to be compact. The missing 'cycle' is then a curve defined in terms of a cutoff regulator. This cutoff construction will be recalled below for the simple case of a local conifold.

JHEP04(2003)028

happens if one picks a geometric regularization instead. In particular, it could prove convenient for some applications to use a regularization which preserves the $\mathrm{SL}(2,\mathbb{Z})$ invariance of the flux superpotential (1.4).

This is the question we wish to study in the present note. The geometric regularization we shall choose will compactify the B-cycles while promoting (1.6) to a meromorphic differential of the second kind on the modified algebraic curve. The result will be a closed Riemann surface, whose periods can be computed in standard manner with the help of Picard-Fuchs equations.

The simplest regularization satisfying our requirements is a small deformation of the fibered geometry (1.2) which transforms it into a monodromic fibration in the sense of [10]. This fibration will have supplementary conifold degenerations when compared with the original geometry. While the resolutions of these singular limits need not admit a simple interpretation in terms of partially wrapped $D5$-branes, the modified geometry does make perfect sense as a string background. On the deformation side, this is still a non-compact background with fluxes, thereby leading to the well-known flux superpotential of [4]–[7]. One can take the limit $\alpha' \to 0$ while keeping the geometric regulator fixed. This leads to an effective four-dimensional description which inherits the $\mathrm{SL}(2,\mathbb{Z})$ duality of the the original type IIB string, a property which is reflected in the manifestly $\mathrm{SL}(2,\mathbb{Z})$ invariant form of the flux superpotential. As in [1], one can decouple gravity by focusing on *one* Calabi-Yau singularization and identifying its vanishing periods with the relevant gaugino condensates. Assuming that there exists an $\mathrm{SL}(2,\mathbb{Z})$ transformation which makes all vanishing cycles carry RR flux, one then takes the limit in which these fluxes N_i become large, while the string coupling computed in that $\mathrm{SL}(2,\mathbb{Z})$ frame vanishes such that the quantities $g_s N_i$ stay fixed.

In this note, we shall focus on the simplest situation, namely when the gauge group remains unbroken, the superpotential is quadratic in Φ and the algebraic curve (1.5) has a single cut in the x plane. Modifying the geometry will lead to a smooth complex torus. We compute the periods by solving the associated Picard-Fuchs equation and give a discussion of the physics that emerges when instead of the A cycle, it is the B cycle which becomes small. We point out some possible generalizations in the last section.

2. The cutoff regularization

The non-compact Calabi-Yau threefold corresponding to the simplest gauge theory vacuum with unbroken $\mathrm{U}(N)$ gauge group is the local conifold:

$$x^2 + y^2 + s^2 + t^2 + \mu = 0\,. \tag{2.1}$$

After integrating over the (s,t) coordinates, this leads to the algebraic curve:

$$y^2 + x^2 + \mu = 0\,. \tag{2.2}$$

Projectivizing (2.2) gives a hyperelliptic curve C in $\mathbb{P}^2$, described by the equation:

$$Y^2 + X^2 + \mu Z^2 = 0\,, \tag{2.3}$$

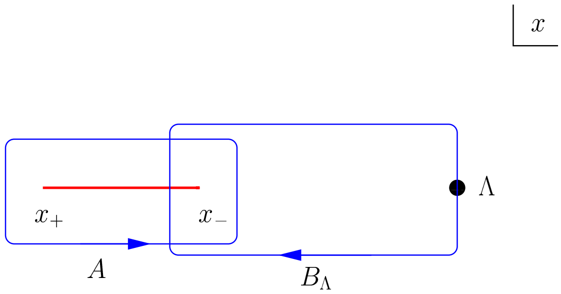

Figure 1: Branch-cut for the undeformed curve.

where X, Y, Z are the homogeneous coordinates. The projective curve (2.3) develops an ordinary double point at the origin for $\mu = 0$. For $\mu \neq 0$, this curve is a smooth Riemann surface of genus *zero*, i.e. a copy of $\mathbb{P}^1$. The quantity y has a single cut which connects the points $x_\pm = \pm i\sqrt{\mu}$ (figure 1).

The projectivized curve has two points above $x = \infty$, which are obtained by setting $Z = 0$ in its defining equation. These are the points $p_\infty^\pm = [1, \pm 1, 0] \in \mathbb{P}^2$, each sitting on one of the branches of (2.3). The cutoff regularization of [2] replaces these with two points sitting at a finite distance along the x-plane. Let us give a precise description of this regularization. Picking a complex number Λ_0 (with $|\Lambda_0| \gg 1$), the curve (2.3) has two points sitting above $x = \Lambda_0$, namely $p_{\Lambda_0}^\pm = [\Lambda_0, \pm i\sqrt{\Lambda_0^2 + \mu^2}, 1]$. Removing these from the projectivized curve gives a twice punctured sphere $\tilde{C} = C - \{p_{\Lambda_0}^+, p_{\Lambda_0}^-\}$, which is conformally equivalent with an infinite cylinder (figure 2). The geometric regularization of [2] amounts to working with this twice-punctured sphere instead of the curve (2.3). The generator of $\pi_1(\tilde{C}) = \mathbb{Z}$ plays the role of A-cycle, while the 'B-cycle' B_{Λ_0} of [2] is an open *path* connecting the points $p^\pm(\Lambda_0)$ sitting in the conformal compactification C of $\tilde{C}$. Hence the regularized B-period $\int_{B_{\Lambda_0}} \omega$ of [2] is a sort of 'holomorphic length' of the cylinder $\tilde{C}$.

In projective coordinates, the differential ω takes the form:

$$\omega = \frac{i}{2}\frac{Y}{Z}\, d\left(\frac{X}{Z}\right) = \frac{i}{2}\left(\frac{1}{Z^2} Y\, dX - \frac{1}{Z^3} XY\, dZ\right). \tag{2.4}$$

To study the behavior at infinity we can go to the coordinate patch $X = 1$ where the curve takes the form $Y^2 + \mu Z^2 + 1 = 0$. For small Z, we find:

$$\omega = -\frac{dZ}{2Z^3}\sqrt{1 + \mu Z^2} = -\frac{dZ}{2Z^3} - \frac{\mu}{4}\frac{dZ}{Z} + O(Z)\, dZ\,, \tag{2.5}$$

which makes the pole with residue $-\mu/4$ explicit. The A-type period can in this case be simply computed as the negative of the residue of ω at $Z = 0$.

3. Geometric regularization of the deformed conifold

We shall replace (2.2) with the 'regularized' curve:

$$y^2 + \epsilon x^3 + x^2 + \mu = 0\,, \tag{3.1}$$

JHEP04(2003)028

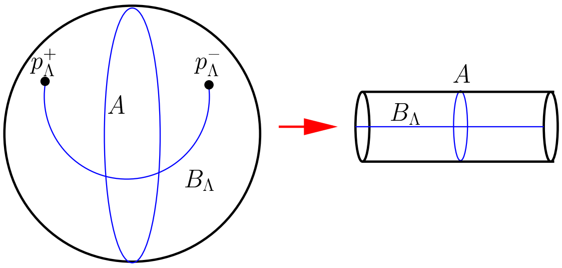

Figure 2: Geometric interpretation of the cutoff regularization.

where we take ϵ to be a small complex quantity. Correspondingly, we replace the local conifold (2.1) with the affine Calabi-Yau threefold:

$$\epsilon x^3 + x^2 + y^2 + s^2 + t^2 + \mu = 0\,. \tag{3.2}$$

This can be viewed as a monodromic A_1 fibration over the x-plane, in the sense of [10]. As in [8, 2, 9, 10], one can integrate the holomorphic 3-form:

$$\Omega = \frac{i}{2\pi}\frac{dx\wedge dy\wedge ds}{t} = -\frac{i}{2\pi}\frac{dx\wedge dy\wedge dt}{s} = \frac{i}{2\pi}\frac{dx\wedge ds\wedge dt}{y} = -\frac{i}{2\pi}\frac{2dy\wedge ds\wedge dt}{3\epsilon x^2+2x} \tag{3.3}$$

over the fiber coordinates s, t in order to reduce it to the meromorphic 1-form $\omega = \frac{i}{2}ydz$ on the Riemann surface (3.1). This is achieved by choosing 3-cycles which are obtained by fibering certain two-spheres sitting inside the s, t fibers over a curve in the x plane.

For $|\epsilon^2\mu| \ll 1$, the x-polynomial in (3.1) has three zeroes, namely:

$$\begin{aligned} x_1 &= x_+ + \frac{1}{2}\mu\epsilon + O(\epsilon^2\mu) \\ x_2 &= x_- + \frac{1}{2}\mu\epsilon + O(\epsilon^2\mu) \\ x_3 &= -\frac{1}{\epsilon} + O(1)\,. \end{aligned} \tag{3.4}$$

Hence the geometric regularization introduces a new cut connecting x_3 and the point at infinity, while performing a small displacement of the cut of the original curve (2.2). In particular, we now have a compact B-cycle encircling x_2 and x_3. This situation is shown in figure 3.

We next consider the projectivization of (3.1), which has the form:

$$Y^2Z + \epsilon X^3 + X^2Z + \mu Z^3 = 0\,. \tag{3.5}$$

This projective curve has genus 1, hence it describes a complex torus (figure 4). It develops singularities for $\mu = 0$ (when the cycle A collapses to an ordinary double point, sitting at $[0, 0, 1]$) and $\mu = -\frac{4}{27\epsilon^2}$ (when the B-cycle is pinched to an ODP sitting at $[-\frac{2}{3\epsilon}, 0, 1]$). The curve (3.5) has a single point sitting above $x = \infty$, namely $p_\infty = [0, 1, 0]$.

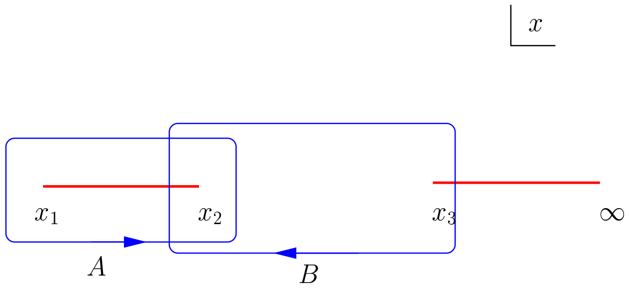

Figure 3: Cuts of the geometrically regularized curve for $|\epsilon^2\mu| \ll 1$.

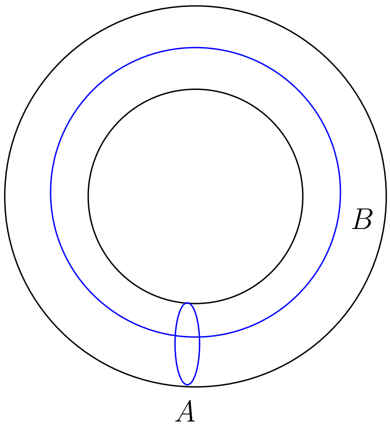

Figure 4: The geometric regularization produces a complex torus.

3.1 The dual degenerations

Let us give a more detailed discussion of the degenerations of (3.1) for $\mu = 0$ and $\mu = -\frac{4}{27\epsilon^2}$. Consider the polynomial:

$$p(x) = \epsilon x^3 + x^2 + \mu \,. \tag{3.6}$$

For $\mu = 0$, this factors as $p(x) = \epsilon x^2(x + 1/\epsilon)$. Hence the roots x_1 and x_2 coalesce in this limit, which means that the cut $[x_1, x_2]$ reduces to a double point (figure 5). The degenerate Riemann surface has branches:

$$y_\pm(x) = \pm i x\sqrt{\epsilon x + 1}\,, \tag{3.7}$$

which are interchanged by the monodromy around x_3.

For $\mu = -\frac{4}{27\epsilon^2}$, we have $p(x) = \epsilon(x - \frac{1}{3\epsilon})(x + \frac{2}{3\epsilon})^2$. In this case, x_2 and x_3 have coalesced to a double point, while x_1 is connected to ∞ by a branch cut (see figure 5). The degenerate surface has branches:

$$y_\pm(x) = \pm i(x + \frac{2}{3\epsilon})\sqrt{\epsilon x - \frac{1}{3}}\,. \tag{3.8}$$

These are interchanged by the monodromy around x_1.

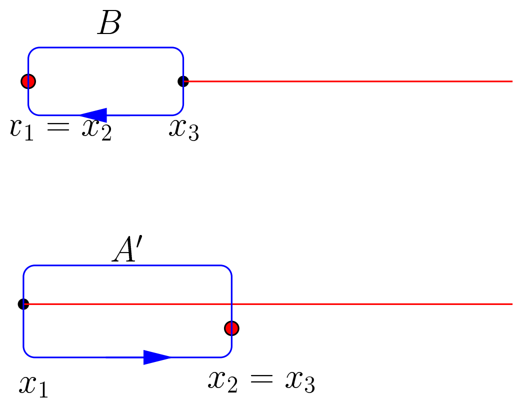

Figure 5: The degenerations $\mu = 0$ (above) and $\mu = -\frac{4}{27\epsilon^2}$ (below). In the second figure, we have slightly displaced the point $x_2 = x_3$ for clarity.

Also notice that the second degenerate curve can be obtained from the first by performing the transformation:

$$\begin{aligned} x &\to -x - \frac{2}{3\epsilon} \\ y &\to -iy\,, \end{aligned} \tag{3.9}$$

which clearly maps (3.7) into (3.8). This change of coordinates maps the ODP of the first degenerate curve into that of the second curve, while interchanging the cuts. Thus (3.9) identifies the two degenerations, while mapping the B cycle of the first into the A cycle of the second. The B-period $\frac{i}{2}\int_B \frac{dx}{2\pi i}y$ of the first degeneration is then mapped to $-i$ times the A-period $\frac{i}{2}\int_{A'} \frac{dx}{2\pi i}y$ of the second degeneration. As we shall see in more detail below, this symmetry can be viewed as a remnant of the $\mathrm{SL}(2,\mathbb{Z})$ symmetry of the type IIB flux background on the geometrically regularized space (3.2).

For each conifold singularization of (3.2), one can see that the two-sphere obtained by a small resolution will not be monodromy invariant. As explained in [10], this prevents us from wrapping D-branes on such a sphere in the resolved geometry, which means that the low energy limit of the type IIB background on the deformed space (3.2) does not admit a simple gauge theory description. To recover a standard gauge-theoretic interpretation, one must take the limit $\epsilon \to 0$. This is exactly what one expects based on the arguments of [5]. Keeping μ finite and small, the limit $\epsilon \to 0$ has the effect of pushing the branch point x_3 toward infinity, thereby replacing the cut $[x_3, \infty]$ with an ordinary double point at $x = \infty$. In this limit, the conifold point at $\mu = -\frac{4}{27\epsilon^2}$ is pushed to infinity in the moduli space. The regularized curve (3.1) tends to the the original curve (2.2) uniformly over compact domains in the x-plane. However we note that, starting with the regularized model, one can take a different limit, namely $\epsilon \to 0$ while $\epsilon^2\mu$ is kept fixed and such that $|1 + \frac{27\epsilon^2\mu}{4}|$ is small. As we shall see below, this is equivalent with the previous limit through an $\mathrm{SL}(2,\mathbb{Z})$ transformation.

The differential ω on the regularized curve (3.5) is given by (2.4). In the patch $X = 1$ we now have a branch point at $Z = 0$ and for this reason we introduce the coordinate

$\zeta^2 = Z$ which is single valued around this point. Expanding ω for small ζ gives:

$$\omega = -\sqrt{\epsilon}\frac{d\zeta}{\zeta^6} - \frac{\mu}{2\sqrt{\epsilon}}\frac{d\zeta}{\zeta^4} + \frac{\mu^2}{8\epsilon^{3/2}}\frac{d\zeta}{\zeta^2} + O(\zeta^0)d\zeta\,, \tag{3.10}$$

showing that ω is an abelian differential of the second kind on (3.5).

4. Periods of the geometrically regularized surface

In this section, we shall extract the (A, B) periods of the meromorphic form $\omega = \frac{i}{2}ydz$ for the regularized curve (3.1). Since the latter is a closed Riemann surface, one can use standard techniques in order to write down a Picard-Fuchs equation for the periods, and extract their moduli dependence by solving this equation.

4.1 The Picard-Fuchs equation

Let us introduce the rescaled quantities:

$$x = \frac{z}{\epsilon}\,, \qquad y = \frac{w}{\epsilon}\,, \qquad \mu = -\frac{4}{27\epsilon^2}\nu\,. \tag{4.1}$$

In terms of these variables, the defining equation (3.1) becomes:

$$w^2 + z^3 + z^2 - \frac{4}{27}\nu = 0\,. \tag{4.2}$$

The discriminant of the polynomial $p(z) = z^3 + z^2 - \frac{4}{27}\nu$ takes the form:

$$\Delta = -\frac{16}{27}\nu(\nu - 1)\,. \tag{4.3}$$

For $\nu = 0$, the elliptic curve (4.2) acquires an ordinary double point (ODP) at the origin, while for $\nu = 1$ it develops an ODP at $z = -2/3$ and $w = 0$. These correspond to the degenerations discussed in the previous section.

Under the redefinitions (4.1), the form $\omega = \frac{i}{2}ydx$ scales as:

$$\omega = \frac{1}{\epsilon^2}\kappa\,, \tag{4.4}$$

where $\kappa := \frac{i}{2}wdz$. This is a meromorphic differential of the second kind on the complex torus (4.2). Thus its periods $U = \int_{\mathcal{C}}\kappa$ (with $\mathcal{C}$ a cycle on the torus) satisfy a Picard-Fuchs equation, which can be extracted with the methods of [19] (see [20] for a systematic approach which can be easily coded):

$$\nu(\nu - 1)\frac{d^2U}{d\nu^2} + \frac{5}{36}U = 0\,. \tag{4.5}$$

Introducing the logarithmic derivative $\delta = \nu\frac{d}{d\nu}$, one can write this in the form:

$$\left[\delta\,(\delta - 1) - \nu\left(\delta - \frac{1}{6}\right)\left(\delta - \frac{5}{6}\right)\right]U = 0\,, \tag{4.6}$$

which can be recognized as a hypergeometric equation with symbol $\begin{bmatrix} -1/6 & & -5/6 \\ & 0 & \end{bmatrix}$.

JHEP04(2003)028

4.2 A period basis

To extract a basis of periods, we shall use the Meijer function technique described in [21, 22]. As explained in that reference, a basis of solutions of (4.5) is provided by the following functions, which we write in terms of their Mellin-Barnes representations:

$$
\begin{aligned}
U_1(\nu) &= \frac{1}{2\pi i\Gamma(-1/6)\Gamma(-5/6)}\int_{\gamma_1}\frac{\Gamma(-s)\Gamma(s-1/6)\Gamma(s-5/6)}{\Gamma(s)}(-\nu)^s \\
U_2(\nu) &= \frac{1}{2\pi i\Gamma(-1/6)\Gamma(-5/6)}\int_{\gamma_2}\frac{\Gamma(1-s)\Gamma(-s)\Gamma(s-1/6)\Gamma(s-5/6)}{\Gamma(s)}\nu^s\,.
\end{aligned}
\tag{4.7}
$$

Here γ_j are contours connecting $-i\infty$ and $+i\infty$ while separating (A) and (B)-type poles of the corresponding integrands. For U_1, the (A)-poles are $s=n$ from $\Gamma(-s)$, while the (B)-poles are $s=-n+1/6$ and $-n+5/6$ from $\Gamma(s-1/6)$ and $\Gamma(s-5/6)$, where n is a non-negative integer. For U_2, the (A)-poles are $s=n$, while the (B)-poles are $s=1/6-n$ and $s=5/6-n$. All poles are simple except for the nonzero (A)-poles of the U_2-integrand, which are double.

For $|\nu|<1$, one can close the contour γ_1 toward $+\infty$ to find:

$$
U_1(\nu)=\frac{5}{36}\nu\ {}_2F_1\left(\begin{smallmatrix}1/6 & & 5/6\\ & 2 & \end{smallmatrix}\middle|\nu\right)=\nu\frac{d}{d\nu}\ {}_2F_1\left(\begin{smallmatrix}-1/6 & & -5/6\\ & 1 & \end{smallmatrix}\middle|\nu\right)\underset{(|\nu|<1)}{=}\sum_{n\geq 1}\frac{(-1/6)_n(-5/6)_n}{n!^2}n\nu^n\,.
\tag{4.8}
$$

Closing γ_2 toward $+\infty$ gives:

$$
U_2(\nu)\underset{(|\nu|<1)}{=}1+U_1(\nu)\ln\nu+\Phi(\nu)\,,
\tag{4.9}
$$

where:

$$
\Phi(\nu)=\sum_{n\geq 1}\frac{(-1/6)_n(-5/6)_n}{n!^2}n\nu^n\left[\psi\left(n-\frac{1}{6}\right)+\psi\left(n-\frac{5}{6}\right)-2\psi(1)+\frac{1}{n}-2\sum_{j=1}^{n}\frac{1}{j}\right].
\tag{4.10}
$$

Here $\psi(z):=\frac{d}{dz}\ln\Gamma(z)$ is the logarithmic derivative of the Γ function.

The expansions of U_1 and U_2 for $|\nu|>1$ can be obtained by closing the contours γ_j toward $-\infty$. This gives:

$$
\begin{aligned}
U_1(\nu) &= e^{\frac{i\pi}{6}}\Phi_1(\nu)-e^{-\frac{i\pi}{6}}\Phi_2(\nu) \\
U_2(\nu) &= \frac{\pi}{\sin\frac{\pi}{6}}\left[\Phi_1(\nu)+\Phi_2(\nu)\right]\,,
\end{aligned}
\tag{4.11}
$$

where:

$$
\begin{aligned}
\Phi_1(\nu) &:= \frac{5}{24}\frac{\sqrt{3}}{\pi}\frac{\Gamma(5/6)^2}{\Gamma(2/3)}\nu^{1/6}\ {}_2F_1\left(\begin{smallmatrix}-1/6 & & 5/6\\ & 5/3 & \end{smallmatrix}\middle|\tfrac{1}{\nu}\right) \\
&= \frac{1}{\Gamma(-1/6)\Gamma(-5/6)}\sum_{n\geq 0}\frac{\Gamma(n-1/6)\Gamma(-n-2/3)}{n!\Gamma(-n+1/6)}\nu^{-n+1/6}
\end{aligned}
\tag{4.12}
$$

$$
\begin{aligned}
\Phi_2(\nu) &:= -\frac{1}{6}\frac{\Gamma(2/3)}{\Gamma(5/6)^2}\nu^{5/6}\ {}_2F_1\left(\begin{smallmatrix}1/6 & & -5/6\\ & 1/3 & \end{smallmatrix}\middle|\frac{1}{\nu}\right) \\
&= \frac{1}{\Gamma(-1/6)\Gamma(-5/6)}\sum_{n\geq 0}\frac{\Gamma(n-5/6)\Gamma(-n+2/3)}{n!\Gamma(-n+5/6)}\nu^{-n+5/6}\,.
\end{aligned}
\tag{4.13}
$$

To find the expansions for $|1-\nu| < 1$, we first notice that the Picard-Fuchs equation (4.5) admits the symmetry:

$$\nu \to 1-\nu\,. \tag{4.14}$$

Defining $U(\nu) := \begin{bmatrix} U_1(\nu) \\ U_2(\nu) \end{bmatrix}$, we must therefore have:

$$U(1-\nu) = JU(\nu)\,, \tag{4.15}$$

for some constant involutive matrix J. Direct computation easily gives:

$$J = \begin{bmatrix} 0 & 1/2\pi \\ 2\pi & 0 \end{bmatrix}, \tag{4.16}$$

which indeed satisfies $J^2 = Id$. Together with (4.15) and (4.8) and (4.9), this specifies the expansions of U_j for $|1-\nu| < 1$. The symmetry (4.14) interchanges the two singular points $\nu = 0$ and $\nu = 1$. This corresponds to the isomorphism (3.9) between the two degenerations of the regularized curve (3.1).

4.3 Monodromies in the Meijer basis

Let us define monodromy matrices around $\nu = 0$ and $\nu = \infty$ by the relations:

$$\begin{aligned} U(e^{2\pi i}\nu) &= T[0]U(\nu) \quad \text{for} \quad |\nu| \ll 1 \\ U(e^{2\pi i}\nu) &= T[\infty]U(\nu) \quad \text{for} \quad |\nu| \gg 1\,. \end{aligned} \tag{4.17}$$

With a similar definition of the monodromy matrix $T[1]$ around $\nu = 1$, we have:

$$T[\infty] = T[1]T[0]\,, \tag{4.18}$$

which results from a similar relation in the fundamental group of the moduli space $\mathcal{M} = \mathbb{P}^1 - \{0, 1, \infty\}$. Using the results of the previous subsection, one easily computes:

$$T[0] = \begin{bmatrix} 1 & 0 \\ 2\pi i & 1 \end{bmatrix}, \qquad T[1] = \begin{bmatrix} 1 & i/2\pi \\ 0 & 1 \end{bmatrix}, \qquad T[\infty] = \begin{bmatrix} 0 & i/2\pi \\ 2\pi i & 1 \end{bmatrix}, \tag{4.19}$$

where we used (4.18). Also notice the relation:

$$T[1] = JT[0]J\,, \tag{4.20}$$

which holds as a consequence of (4.15).

4.4 The integral basis

Choosing the canonical basis (A, B) as above, we define the periods:

$$S = \frac{i}{2}\int_A \frac{dx}{2\pi i} y \qquad \text{and} \qquad \Pi = \frac{i}{2}\int_B \frac{dx}{2\pi i} y\,. \tag{4.21}$$

It is easy to see that these are related to the Meijer periods by the rescalings:

$$\begin{aligned} S &= \frac{36}{5\times 27}\frac{U_1}{\epsilon^2} \\ \Pi &= \frac{1}{2\pi i}\frac{36}{5\times 27}\frac{U_2}{\epsilon^2}\,. \end{aligned} \tag{4.22}$$

In the canonical basis, the monodromies take the form:

$$\tilde{T}[0] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad \tilde{T}[1] = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \qquad \tilde{T}[\infty] = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, \tag{4.23}$$

while the matrix J is replaced by:

$$\tilde{J} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \tag{4.24}$$

The first two monodromies have the Picard-Lefschetz form. In the limit $\nu = \infty$, both periods of the elliptic curve blow up.

4.5 The flux superpotential

Given the periods (4.22), one can now compute the flux superpotential of [2]

$$W_{\text{eff}} = -2\pi i\,[N\Pi + \alpha S] \tag{4.25}$$

everywhere on the moduli space of the geometrically regularized model. Of this moduli space, the regions of interest for the field theory applications are the vicinities of the two conifold points. We now show how the leading contributions to the flux superpotential around these points can be recovered from the periods (4.22), thereby yielding the Veneziano-Yankielowicz superpotential in our regularization.

4.5.1 The flux superpotential for $|\nu| \ll 1$

For small ν, we have:

$$\begin{aligned} U_1 &= \frac{5}{36}\nu + O(\nu^2) \\ U_2 &= 1 + \frac{5}{36}\nu\,[\ln\nu - 1 - \ln(16\times 27)] + O(\nu^2)\,, \end{aligned} \tag{4.26}$$

where we used the identity:

$$\psi\left(\frac{1}{6}\right) + \psi\left(\frac{5}{6}\right) - 2\psi(1) = -\log(16\times 27)\,. \tag{4.27}$$

This gives:

$$\begin{aligned} S &\approx \frac{\mu}{4} \\ \Pi &\approx \frac{1}{2\pi i}\left[\frac{36}{5\times 27}\frac{1}{\epsilon^2} + S\left(\ln\frac{\epsilon^2}{16} + \ln S - 1\right)\right]. \end{aligned} \tag{4.28}$$

Defining Λ_0 and Λ through:

$$\begin{aligned}\Lambda_0^3 &= \frac{16}{\epsilon^2}\\ \Lambda^{3N} &= \Lambda_0^{3N} e^{-2\pi i\alpha}\,,\end{aligned} \tag{4.29}$$

we easily obtain:

$$W_{\text{eff}} = -\frac{N\Lambda_0^3}{60} + W_{VY}(S,\Lambda) + O\left(\frac{1}{\Lambda}\right), \tag{4.30}$$

where:

$$W_{VY}(S,\Lambda) = S\ln\frac{\Lambda^{3N}}{S^N} + NS \tag{4.31}$$

is the Veneziano-Yankielowicz superpotential.

4.5.2 The flux superpotential for $|1-\nu| \ll 1$

This results immediately from the above upon using the symmetry (4.15). Defining:

$$\hat{N} = -\alpha\,, \qquad \hat{\alpha} = +N\,, \tag{4.32}$$

we have:

$$W_{eff} = \frac{\hat{N}\hat{\Lambda}_0^3}{60} + W_{VY}(\Pi,\hat{\Lambda}) + O\left(\frac{1}{\hat{\Lambda}}\right), \tag{4.33}$$

where:

$$\hat{\Lambda}^{3\hat{N}} = \hat{\Lambda}_0^{3\hat{N}} e^{-2\pi i\hat{\alpha}}\,, \tag{4.34}$$

with:

$$\hat{\Lambda}_0^3 = -\frac{16i}{\epsilon^2}\,. \tag{4.35}$$

Relation (4.32) corresponds to the $\mathrm{SL}(2,\mathbb{Z})$ transformation:

$$\begin{bmatrix}\hat{\alpha}\\ \hat{N}\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}\alpha\\ N\end{bmatrix}, \tag{4.36}$$

which accompanies the transformation (4.24) on the periods S,Π. This agrees with the discussion in subsection 3.1.

5. Summary and outlook

By considering a geometric regularization of the local conifold, we obtained a set of periods defined in terms of compact cycles. The regularization preserves the $\mathrm{SL}(2,\mathbb{Z})$ invariance of the flux superpotential and allows one to determine this quantity at every point on the moduli space by making use of the standard technique of Picard-Fuchs equations.

The regularized geometry (3.2) can be viewed as a monodromic A_1 fibration in the sense of [10]. This means that the resulting string background cannot be produced by a geometric transition from a background with wrapped $D5$-branes. Indeed, turning off the deformation parameter μ one finds a local conifold degeneration at $x = y = s = t = 0$. One could blow up this point to a two-sphere. However, there is a non-trivial monodromy around the point $x_3 = -\frac{1}{\epsilon}$ produced by the regulator ϵ, and going around this point in the x-plane takes the two sphere to minus itself. As explained in [10], D-branes can only be wrapped on monodromy-invariant cycles of a fibered ADE-geometry.

However, the regularized geometry is a valid flux background of IIB string theory and as such it leads to a non-trivial $\mathcal{N} = 1$ superpotential in four dimensions. This flux superpotential is explicitly $\mathrm{SL}(2,\mathbb{Z})$ invariant. An interesting feature of this construction is that the regularized space admits two conifold degenerations, which are achieved for different values of the deformation parameter μ. At the first conifold point (which occurs for $\mu = 0$), the A-cycle vanishes. There one can perform a double scaling limit which recovers the usual geometry of [1, 2]. From this point of view the effective four dimensional description which arises in the limit $\alpha' \to 0$ of the IIB string theory serves as a sort of $\mathrm{SL}(2,\mathbb{Z})$ invariant completion of the strongly coupled $\mathcal{N} = 1$ gauge theory. The superpotential (1.4) is only the genus zero part in the genus expansion. In order to decouple higher genus contributions, which correspond to gravitational corrections, one can take $N \to \infty$ while keeping Ng_s fixed, where g_s is the string coupling in the $\mathrm{SL}(2,\mathbb{Z})$ frame where the A-cycle carries RR flux.

As we have seen, the regularized geometry has a second conifold degeneration for $\mu = -\frac{4}{27\epsilon^2}$. There it is the B-cycle (the cycle carrying NS-NS flux) which shrinks to zero size. By explicitly calculating the periods, we found that mathematically this second conifold point is completely equivalent to the first. Physically, this can be understood by performing an $\mathrm{SL}(2,\mathbb{Z})$ duality transformation of the IIB string, which exchanges the RR and NS-NS sectors, while inverting the string coupling $g_s \to g_s' = \frac{1}{g_s}$. One can then go trough the same steps as before, by defining $\hat{N} = -\alpha$ and $\hat{\alpha} = N$ and performing a double scaling limit which keeps the B-period finite while taking $\hat{N} \to \infty$ with $\hat{N} g_s'$ fixed. From the point of view of the original $\mathrm{SL}(2,\mathbb{Z})$ frame, this is a strong coupling limit. That leading terms in the effective superpotential still take the Veneziano-Yankielowicz form in this limit is not surprising, since as a holomorphic quantity it is protected and thus can be computed at strong string coupling with the help of NS-NS-flux instead of RR-flux. Hence the geometry (3.2) provides us with a manifestly $\mathrm{SL}(2,\mathbb{Z})$ invariant flux background.

It is straightforward to extend the geometric regularization to more complicated cases, e.g. for multi cut situations corresponding to a breaking of the gauge group to n factor subgroups, which are engineered by the space (1.2). In this case, the regularization replaces (1.2) with:

$$\epsilon x^{2n+1} + W'(x)^2 + y^2 + t^2 + s^2 = 0\,. \tag{5.1}$$

Reducing this in the manner of [9]–[8] and [2], gives a genus n hyperelliptic Riemann surface, thus introducing a new branch point and a new cut connecting it with the point at infinity. Again the reduction ω of the holomorphic three-from of (5.1) produces a meromorphic differential of the second kind. There will be an $Sp(2n,\mathbb{Z})$ symmetry acting on a symplectic basis of A and B cycles which together with type IIB $\mathrm{SL}(2,\mathbb{Z})$ duality transformations should give rise to various dual superpotentials. Another possibility is that two or more cycles with non-vanishing intersection form vanish at a point at finite distance in the moduli space. It would be interesting to study if this happens and what the physical interpretation could be. Similar methods could also be applied for generalizations based on ADE fibrations [9, 10, 24] or orientifolds [23]. Finally, let us mention that the geometric regularization could prove useful in the study of flux backgrounds with orientifolds [25], where one also encounters ‘noncompact cycles’ when using a cutoff regularization. Since the geometric regularization allows one to apply standard Picard-Fuchs techniques, it could also be useful for explicit computations of periods in a large number of situations involving non-compact Calabi-Yau spaces.

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JHEP04(2003)028